Problem 3.23

Show that projection operators are **idempotent**: $\hat{P}^2 = \hat{P}$. Determine the eigenvalues of $\hat{P}$, and characterize its eigenvectors.

Solution

For some vector $|\alpha\rangle$, the operator

$$\hat{P} = |\alpha\rangle\langle\alpha|$$

can be formed. In order for this to be a projection operator, $|\alpha\rangle$ must be normalized ($\langle\alpha\,|\,\alpha\rangle = 1$) so that

$$\begin{aligned}
\hat{P}^2 &= \hat{P}\hat{P} \\
&= (|\alpha\rangle\langle\alpha|)\,(|\alpha\rangle\langle\alpha|) \\
&= |\alpha\rangle\langle\alpha\,|\,\alpha\rangle\langle\alpha| \\
&= |\alpha\rangle(1)\langle\alpha| \\
&= |\alpha\rangle\langle\alpha| \\
&= \hat{P}.
\end{aligned}$$

Consider the eigenvalue problem for $\hat{P}$.

$$\hat{P}|f\rangle = p|f\rangle \tag{1}$$

Apply the projection operator to both sides.

$$\hat{P}\left(\hat{P}|f\rangle\right) = \hat{P}\,(p|f\rangle)$$

$$\hat{P}^2|f\rangle = p\left(\hat{P}|f\rangle\right)$$

$$\hat{P}^2|f\rangle = p\,(p|f\rangle)$$

$$\hat{P}^2|f\rangle = p^2|f\rangle$$

Use the fact that $\hat{P}^2 = \hat{P}$.

$$\hat{P}|f\rangle = p^2|f\rangle \tag{2}$$

Subtract the respective sides of equations (2) and (1).

$$\hat{P}|f\rangle - \hat{P}|f\rangle = p^2|f\rangle - p|f\rangle$$

$$|0\rangle = (p^2 - p)|f\rangle$$

Since the eigenvector $|f\rangle$ can't be the zero vector, it must be the case that

$$p^2 - p = 0$$

$$p(p-1) = 0$$

$$p = \{0, 1\}.$$

These are the eigenvalues of $\hat{P}$.

Now plug each of them into equation (1) to determine the corresponding eigenvectors.

$$\hat{P}|f\rangle = 0|f\rangle \qquad \hat{P}|f\rangle = 1|f\rangle$$

$$\hat{P}|f\rangle = |0\rangle \qquad \hat{P}|f\rangle = |f\rangle$$

$$(|\alpha\rangle\langle\alpha|)\,|f\rangle = |0\rangle \qquad (|\alpha\rangle\langle\alpha|)\,|f\rangle = |f\rangle$$

$$|\alpha\rangle\langle\alpha\,|\,f\rangle = |0\rangle \qquad |\alpha\rangle\langle\alpha\,|\,f\rangle = |f\rangle$$

This equation on the left is only satisfied if $\langle\alpha\,|\,f\rangle = 0$, and this equation on the right is only satisfied if $|f\rangle$ is some multiple of $|\alpha\rangle$, that is, $|f\rangle = c\,|\alpha\rangle$.

$$|\alpha\rangle(0) = |0\rangle \qquad |\alpha\rangle\langle\alpha|\,(c\,|\alpha\rangle) = c\,|\alpha\rangle$$

$$|0\rangle = |0\rangle \qquad c\,|\alpha\rangle\langle\alpha\,|\,\alpha\rangle = c\,|\alpha\rangle$$

$$c\,|\alpha\rangle(1) = c\,|\alpha\rangle$$

$$c\,|\alpha\rangle = c\,|\alpha\rangle$$

Therefore, the eigenvector associated with $p = 0$ is orthogonal to $|\alpha\rangle$, and the eigenvector associated with $p = 1$ is parallel to $|\alpha\rangle$.